The Ginzburg criterion

The solution of the Landau approximation of the free-energy (corresponding to Ginzburg - Landau without gradient term) has the mean-field critical exponents, that do not correspond to the correct ones of the Ising model in 2 and 3 dimensions. This tells us that the gradient terms are necessary.

But their necessity does not imply that they are sufficient. Is there a dimension d such that they are negligible and the exact exponents are the ones of the mean-field?

This is what the Ginzburg criterion is about.

We have to compare the magnetization to its fluctuations. If the magnetization dominates, then mean-field is exact; otherwise it is not.

From the mean-field solution we know that

$$m = a|t|^{1/2} \qquad \text{with } a > 0$$

To compute the fluctuations, we recall that

$$\chi \propto \langle M^2 \rangle - \langle M \rangle^2 = \langle (M - \langle M \rangle)^2 \rangle = \langle \delta M^2 \rangle$$

But we also recall that

$$\chi = \frac{dM}{dh} \propto \left(\sum_i s_i\right)\left(\sum_j s_j\right) = \sum_i \sum_j s_i s_j =$$

$$= \sum_i \sum_k s_i s_{i-k} \quad =$$

here I use a loose definition. It just means that I can run over the lattice using i as an origin

$$= \sum_k \underbrace{\sum_i s_i s_{i-k}}$$

this is nothing else than $c(\vec{r}) \approx a \dfrac{e^{-r/\xi}}{r^{d-2}}$

mean field

$$\Downarrow$$

$$\chi = \langle \delta M^2 \rangle \simeq A \int_0^\infty dr\, r^{d-1} \frac{e^{-r/\xi}}{r^{d-2}} =$$

also contains the angular part

$$= A\, \xi^2 \int_0^\infty d\left(\frac{r}{\xi}\right)\left(\frac{r}{\xi}\right)^{d-1} \frac{e^{-r/\xi}}{\left(\frac{r}{\xi}\right)^{d-2}} =$$

$$= A\, \xi^2 \int_0^\infty dy\, y\, e^{-y} = \tilde{A}\, \xi^2$$

But $\xi \sim |t|^{-1/2}$ mean-field

so that

$$\langle \delta M^2 \rangle \sim \tilde{A}\, t^{-1}$$

But $\langle \delta M^2 \rangle$ is also the fluctuation of the energy due to "correlated" magnetization fluctuations.

The energy is instead:

$$\int m^2 \, d\vec{x} \simeq |t|^{2 \cdot 1/2} \cdot \underbrace{\xi^d}_{} \sim |t|^1 \cdot |t|^{-d/2} = |t|^{1-d/2}$$

↳ fluctuations are correlated within a distance ξ, so we must compare them to the energy due to magnetization in a comparable volume

Mean-field is valid if $\underbrace{|t|^{1-d/2}}_{\text{bulk energy}} \gg \underbrace{\tilde{A}\,|t|^{-1}}_{\text{energy of fluctuations}}$

$$\Rightarrow \quad |t|^{\frac{4-d}{2}} \gg \tilde{A}$$

But $|t| \ll 1$ (infinitesimally close to T_c: $|t| = \left|\frac{T-T_c}{T_c}\right|$)

The inequality can be satisfied only if $\frac{4-d}{2} < 0$

$$\Rightarrow \quad d > 4$$

Upper critical dimension of the Ising model: $\boxed{d=4}$

Above the upper critical dimension, the mean-field exponents are correct